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Sufficient conditions for optimality for differential inclusions of parabolic type and duality

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Abstract Sufficient conditions for optimality are derived for partial differential inclusions of parabolic type on the basis of the apparatus of locally conjugate mapping, and duality theorems are proved. The duality theorems proved allow one to conclude that a sufficient condition for an extremum is an extremal relation for the direct and dual problems.

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1 Introduction

It is known that optimization problems for differential inclusions constitute one of the intensively developing directions in optimal control theory [1–20]. The reason is mainly the fact that a great number of problems in economic dynamics, as well as classical problems on optimal control, differential games, and so on, can be reduced to such investigations [1,18,21,22].

The present article is devoted to an investigation of this kind with first boundary value problems for partial differential inclusions of parabolic type. It can be divided conditionally into two parts. In the first part in terms of locally conjugate mappings (LCM) we formulate both for convex and non-convex problems sufficient conditions for optimality. The proved theorems show that the conjugate differential inclusions have various unexpected form for different type of differential inclusions [5,6,16,20]. Further, the obtained results are generalized to the multi-dimensional case with a second-order elliptic operator for bounded cylindrical domains.

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In the second part of the paper we construct the dual problem to convex problems for partial differential inclusions of parabolic type. As is known, duality theory is by virtue of the importance of its applications one of the central directions in convex optimality problems, and it is interpreted differently for different concrete cases (in mathematical economics, in mechanics, and so on). Besides the indicated applications, duality often makes it possible to simplify the computational procedure and to construct a generalized solution of variational problems that do not have classical solutions.

The duality theorems proved allow one to conclude that a sufficient condition for an extremum is extremal relation for the direct and dual problems. The latter means that if some pair of admissible solutions satisfies this relation, then each of them is a solution of the corresponding (direct or dual) problems. We remark that a significant part of the investigations for simple variational problems and ordinary differential inclusions is connected with such problems [12,14–17,23].

It is known that for the theory of partial differential equations the concept of a generalized solution is important both from the theoretical and from the practical point of view [22,24].

We emphasize that the solution of the considered differential inclusions is taken in the space $C^{1,2}$ [22,25]. However, as will be seen from the context, the definition below of the concept of a solution in this or that sense is introduced only for simplicity and does not in any way restrict the class of problems, under consideration.

As a rule the definition of such solutions associates with a given equation a certain integral identity that uses, in turn, the class of generalized derivatives and compactly supported functions. Therefore, on this path the most natural approach for partial differential inclusions is apparently the use of single-valued branches selections of a multivalued mapping.

At the end of Sect. 4 we consider an optimal control problem described by the heat equation. This example shows that in known problems the conjugate inclusion coincides with the conjugate equation which is traditionally obtained with the help of the Hamiltonian function.

2 Necessary facts and problem statement

The basic concepts and definitions given below can be found in Refs. [1,15–17]. Let R^n be the n-dimensional Euclidean space, (x_1, x_2) is a pair of elements $x_1, x_2 \in R^n$ and $\langle x_1, x_2\rangle$ is their inner product. We say that a multivalued mapping a: $R^n \to 2^{R^n}$ is a convex if its graph $grf a = \{(x, v) : v \in a(x)\}$ is a convex for each $x \in \text{dom } a = \{x : a(x) \neq \emptyset\}$. For such mappings we introduce the notation

$$W_a(x, v^*) = \inf_v \{\langle v, v^*\rangle : v \in a(x)\}, x, v^* \in R^n,$$

$$a(x, v^*) = \{v \in a(x) : \langle v, v^*\rangle = W_a(x, v^*)\}.$$

For convex a we let $W_a(x, v^*) = +\infty$ if $a(x) = \emptyset$.

For a convex mapping a the cone of tangent directions at a point $(x^0, v^0) \in grfa$ will be denoted by $K_a(x^0, v^0)$:

$$\begin{aligned} K_a(x^0, v^0) &= \text{con}(grf a - (x^0, v^0)) \\ &= \{(\overline{x}, \overline{v}) : \overline{x} = \lambda(x - x^0)\overline{v} = \lambda(v - v^0), \lambda > 0, (x, v) \in grfa\}. \end{aligned}$$

Moreover for a convex mapping a and a convex set $M \subset R^n$:

$$\Omega_a(x^*, v^*) = \inf\{-\langle x, x^*\rangle + \langle v, v^*\rangle : (x, v) \in gfa\},$$

$$W_M(v^*) = \inf_{v \in M} \langle v^*, v \rangle.$$

It is obvious that $-W_M(-v^*)$ is a support function to M.

A mapping a^*: $R^n \times R^n \times R^n \to R^n$ defined by

$$a^*(v^*; (x, v)) = \{x^* : (-x^*, v^*) \in K_a^*(x, v)\}$$

is called the LCM to a at the point (x, v), where $K_a^*(x, v)$ is the cone dual to the cone $K_a(x, v)$. Furthermore for a non-convex mapping a (*grf a is nonconvex*) we define the LCM at the point $(\tilde{x}, \tilde{v})$, $\tilde{v} \in a(x, v^*)$ as follows:

$$a^*(v^*; (\tilde{x}, \tilde{v})) = \{x^* : W_a(x, v^*) - W_a(\tilde{x}, v^*) \geq \langle x^*, x - \tilde{x} \rangle, \forall x \in R^n\} \tag{0}$$

For a function g: $R^n \to R^1 \cup \{\pm\infty\}$ we put

$g^*(x^*) = \sup_x \{\langle x, x^* \rangle - g(x)\}$, dom g $= \{x : g(x) < +\infty\}$.

Here g^* is called the conjugate function of a function g. A function is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.

For convex mappings a [1, Theorem 2.1] it holds

$$a^*(v^*; (x, v)) = \left\{ \begin{array}{ll} \partial W_a(x, v^*), & v \in a(x, v^*), \\ \emptyset, & v \notin a(x, v^*) \end{array} \right\},$$

where $\partial W_a(x, v^*)$ is a subdifferential of convex function $W_a(., v^*)$ at a point $x \in \text{dom} W_a(., v^*)$. It follows from the last formula and formula (0) that for convex multi-functions their defined the same LCM.

In Sect. 3 we study the problem for so called partial differential inclusions of parabolic type:

$$I(x) = \iint_Q g(x(t, \tau), t, \tau) \mathrm{d}t \mathrm{d}\tau \to \inf, \tag{1}$$

where

$$\frac{\partial^2 x(t, \tau)}{\partial \tau^2} - \frac{\partial x(t, \tau)}{\partial t} \in a(x(t, \tau), t, \tau), \quad 0 < t \leq 1, \quad 0 < \tau < 1, \tag{2}$$

$$x(0, \tau) = \alpha(\tau), x(t, 0) = \beta_0(t), \quad x(t, 1) = \beta_1(t), \tag{3}$$

$$Q = [0, 1] \times [0, 1].$$

Here $a(., t, \tau)$: $R^n \to 2^{R^n}$ is multivalued mapping, g: $R^n \times Q \to R$ and α and β_0, β_1 are continuous functions, α: $[0, 1] \to R^n$, β_i: $[0, 1] \to R^n (i = 0, 1)$.The problem is to find a classical solution $\widetilde{x}(t,\tau)$ of the so-called first boundary value problem (2) and (3) that minimizes I. Note that in Refs [3,4,6,7,14–17] the classical optimal control problems described by hyperbolic or elliptic type differential equations are extended to the case of corresponding differential inclusions and the obtained problems, naturally are called problems for hyperbolic, elliptic differential inclusions, respectively. Of course, in this sense the name parabolic differential inclusions for considered problems (1)–(3) is justified.

Here a solution is understood to be classical solution only for simplicity of the exposition. As will be seen in Sects 3–4 the results obtained can be introduced to the case of a generalized solution.

The subject of the next investigation in Sect. 3 is the multi-dimensional optimal control problem for partial differential inclusions of parabolic type:

$$I(x) = \int_0^1 \int_G g(x(t,\tau),t,\tau)\mathrm{d}t\mathrm{d}\tau \to \inf, \tag{4}$$

subject to

$$Lx(t,\tau) \in \frac{\partial x(t,\tau)}{\partial t} + a(x(t,\tau)), \tag{5}$$

$$x(0,\tau) = \alpha(\tau), \qquad \tau \in G \subset R^n, \tag{6}$$

$$x(t,\tau) = \beta(t,\tau), \qquad (t,\tau) \in H, \tag{7}$$

where a: $R \to 2^R$, and G is the domain of change of arguments $\tau = (\tau_1,\ldots,\tau_n)$ in the differential inclusion (5), with piecewise smooth boundary S. Thus, the domain in which (5) is given is a cylinder $D = \{\tau \in G, 0 < t < 1\}(D \subset R^{n+1})$ of height 1 and with base G, H is the lateral surface of D: $H = \{\tau \in S, 0 < t < 1\}$, and $G \times \{0\}$ and $G\times\{1\}$ are the lower and upper bases, respectively. Further,

$$Lx = \sum_{i,j=1}^{n} \frac{\partial}{\partial \tau_i}(d_{ij}(\tau)\frac{\partial x}{\partial \tau_j}) + \sum_{i=1}^{n} b_i(\tau)\frac{\partial x}{\partial \tau_i} + c(\tau)x$$

is a second-order elliptic operator, $d_{ij} = d_{ji}$, $i, j = 1,\ldots,n$, and moreover, there exists a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{n} d_{i,j}(\tau)\beta_i\beta_j \geq \gamma \sum_{i=1}^{n} \beta_i^2, \quad \tau \in \overline{G,}$$

for any real numbers $\beta_1,\ldots,\beta_n, \sum_{i=1}^{n}\beta_i^2 \neq 0$, $d_{ij} \in C^1(\overline{D})$, and $b_i \in C^1(\overline{D})$, $c \in C(\overline{D})$are given real functions (however, we can assume without loss of generality that $b_i(\tau) = c(\tau) \equiv 0$). Here $C(\overline{D})$ and $C^1(\overline{D})$ are the spaces of continuous functions and functions having a continuous first order derivatives in D, respectively.

A function $x(t,\tau)$ in $C^{1,2}(D) \cap C[D \cup H \cup (G \times \{0\})]$, that satisfies the inclusion (5) in D, the initial condition (6) on G $x\{0\}$, and the boundary condition (7) on H is called a classical solution of the problem posed, where $C^{1,2}(D)$ is the space of functions u, having continuous derivatives $\partial u/\partial t$, $\partial^2 u/\partial\tau_i\partial\tau_j$, $i, j = 1,\ldots,$ n [22,25].

3 Sufficient conditions for optimality for differential inclusions of parabolic type

Theorem 3.1 *Suppose that g: $R^n \times Q \to R$ is continuous and convex with respect to x and a: $R^n \times Q \to R^n$ is convex mapping. Then for the optimality of the solution $\widetilde{x}(t,\tau)$ among all admissible solutions in the problem (1)–(2) it is sufficient that there exist classical solution $x^*(t,\tau)$ of the following boundary value problem:*

(a) $\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} \in a^*\left(x^*(t,\tau); (\tilde{x}(t,\tau), \tilde{x}''_{\tau^2}(t,\tau) - \tilde{x}'_t(t,\tau)), t, \tau\right) + \partial g(\widetilde{x}(t,\tau),t,\tau),$

(b) $x^*(1,\tau) = 0, \quad x^*(t,0) = 0, \quad x^*(t,1) = 0, \quad t,\tau \in Q,$

(c) $\frac{\partial^2 \widetilde{x}(t,\tau)}{\partial \tau^2} - \frac{\partial \widetilde{x}(t,\tau)}{\partial t} \in a(\widetilde{x}(t,\tau), x^*(t,\tau), t, \tau).$

Proof For convex multivalued mapping a using the Moreau–Rockefeller theorem [1,26,27] from condition (a) we obtain the inclusion

$$\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t}\in\partial[W_a(\widetilde{x}(t,\tau),x^*(t,\tau),t,\tau)+\mathrm{g}(\widetilde{x}(t,\tau),t,\tau)],\quad (t,\tau)\in Q.$$

Using the definitions of subdifferential and condition (c) for all admissible solutions $x(t,\tau)$ we rewrite the last relation in the form

$$\left\langle\frac{\partial^2 x(t,\tau)}{\partial\tau^2}+\frac{\partial x(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle-\left\langle\frac{\partial^2 \widetilde{x}(t,\tau)}{\partial\tau^2}+\frac{\partial \widetilde{x}(t,\tau)}{\partial t},x^*(t,\tau)\right\rangle$$
$$+\mathrm{g}(x(t,\tau),t,\tau)-\mathrm{g}(\widetilde{x}(t,\tau),t,\tau)\geq\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t},x(t,\tau)-\widetilde{x}(t,\tau)\right\rangle.$$

Integrating this inequality over Q, we get

$$\iint_{\mathrm{Q}}[\mathrm{g}(x(t,\tau),t,\tau)-\mathrm{g}(\widetilde{x}(t,\tau),t,\tau)]\mathrm{d}t\mathrm{d}\tau\geq\iint_{\mathrm{Q}}\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2}+\frac{\partial x^*(t,\tau)}{\partial t},$$
$$x(t,\tau)-\widetilde{x}(t,\tau)\rangle\mathrm{d}t\mathrm{d}\tau$$
$$-\iint_{\mathrm{Q}}\langle\frac{\partial^2(x(t,\tau)-\widetilde{x}(t,\tau))}{\partial\tau^2}-\frac{\partial(x(t,\tau)-\widetilde{x}(t,\tau))}{\partial t},x^*(t,\tau)\rangle\mathrm{d}t\mathrm{d}\tau. \tag{8}$$

After simple transformations we obtain that the right side of the inequality (8) is equal to zero. Indeed for brevity of notation we denote the right side of (8) by R. Then

$$R=\iint_{\mathrm{Q}}\left\langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2},x(t,\tau)-\widetilde{x}(t,\tau)\right\rangle\mathrm{d}t\mathrm{d}\tau$$
$$-\iint_{\mathrm{Q}}\left\langle\frac{\partial^2(x(t,\tau)-\widetilde{x}(t,\tau))}{\partial\tau^2},x^*(t,\tau)\right\rangle\mathrm{d}t\mathrm{d}\tau$$
$$+\iint_{\mathrm{Q}}\frac{\partial}{\partial t}\langle x(t,\tau)-\widetilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\mathrm{d}\tau,$$

where, since by (b) $x^*(1,\tau)=0$ and $x(t,\tau)$ and $\widetilde{x}(t,\tau)$ are admissible solutions, that is $x(0,\tau)=\widetilde{x}(0,\tau)=\alpha(\tau)$ we have

$$\iint_{\mathrm{Q}}\frac{\partial}{\partial t}\langle x(t,\tau)-\widetilde{x}(t,\tau),x^*(t,\tau)\rangle\mathrm{d}t\mathrm{d}\tau=\int_0^1\langle x(1,\tau)-\tilde{x}(1,\tau),x^*(1,\tau)\rangle\mathrm{d}\tau$$
$$-\int_0^1\langle x(0,\tau)-\tilde{x}(0,\tau),x^*(0,\tau)\rangle\mathrm{d}\tau=0.$$

So it is clear that

$$R = \iint_Q \frac{\partial}{\partial \tau} \langle \frac{\partial x^*(t,\tau)}{\partial \tau}, x(t,\tau) - \widetilde{x}(t,\tau) \rangle \mathrm{d}t\mathrm{d}\tau$$
$$- \iint_Q \frac{\partial}{\partial \tau} \langle x^*(t,\tau), \frac{\partial}{\partial \tau}(x(t,\tau) - \widetilde{x}(t,\tau)) \rangle \mathrm{d}t\mathrm{d}\tau$$
$$= \int_0^1 \langle \frac{\partial x^*(t,1)}{\partial \tau}, x(t,1) - \widetilde{x}(t,1) \rangle \mathrm{d}t - \int_0^1 \langle \frac{\partial x^*(t,0)}{\partial \tau}, x(t,0) - \widetilde{x}(t,0) \rangle \mathrm{d}t$$
$$- \int_0^1 \langle x^*(t,1), \frac{\partial}{\partial \tau}(x(t,1) - \widetilde{x}(t,1)) \rangle \mathrm{d}t$$
$$+ \int_0^1 \langle x^*(t,0), \frac{\partial}{\partial \tau}(x(t,0) - \widetilde{x}(t,0)) \rangle \mathrm{d}t = 0,$$

where it is taken into account that $x^*(t,0) = x^*(t,1) = 0$ by the condition (b) of the theorem. Thus, we have finally

$$\iint_Q \mathrm{g}(x(t,\tau), t, \tau)\mathrm{d}t\mathrm{d}\tau \geq \iint_Q \mathrm{g}(\widetilde{x}(t,\tau), t, \tau)\mathrm{d}t\mathrm{d}\tau.$$

The theorem is proved.

Theorem 3.2 *Let us consider the non-convex problem (1)–(3) and let $\widetilde{x}(t,\tau)$ be an admissible solution of this problem. Then for the optimality of the solution $\widetilde{x}(t,\tau)$ among all admissible solutions it is sufficient that there exist classical solution $x^*(t,\tau)$ of the following boundary value problem:*

(i) $\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} \in a^* \left(x^*(t,\tau); (\tilde{x}''_{\tau^2}(t,\tau) - \tilde{x}'_t(t,\tau)), t, \tau \right)$,
(ii) $g(x,t,\tau) - g(\widetilde{x}(t,\tau), t, \tau) \geq \langle x^*(t,\tau), x - \widetilde{x}(t,\tau) \rangle$ for all x,
(iii) $x^*(1,\tau) = 0, \quad x^*(t,0) = 0, \quad x^*(t,1) = 0$,
(iv) $\frac{\partial^2 \widetilde{x}(t,\tau)}{\partial \tau^2} - \frac{\partial \widetilde{x}(t,\tau)}{\partial t} \in a(\widetilde{x}(t,\tau), x^*(t,\tau), t, \tau), (t,\tau) \in Q.$

Here the LCM a^* is defined by (0).

Proof By the definition of an LCM a^* and by condition (i), (ii) we have

$$W_a(x, (t,\tau), x^*(t,\tau), t, \tau) - W_a(\widetilde{x}, (t,\tau), x^*(t,\tau), t, \tau) \geq \langle x^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau) \rangle,$$
$$\mathrm{g}(x, (t,\tau), t, \tau) - \mathrm{g}(\widetilde{x}, (t,\tau), t, \tau) \geq \langle x^*(t,\tau), x(t,\tau) - \widetilde{x}(t,\tau) \rangle. \tag{9}$$

Now using the condition (iv) and by integrating the relations (9) and adding them we get the inequality (8). Then starting from the inequality (8) similarly as in the proof of Theorem 3.1 it can be shown that $\widetilde{x}(t,\tau)$ is optimal.

We now try to apply the results of this section to the problem (4)–(7); here the use of Theorem 3.1 plays a decisive role in the investigation of this problem.

Theorem 3.3 *Suppose g is continuous function and convex with respect to x, and a is a convex mapping. Then $\widetilde{x}(t,\tau)$ minimizes the functional (4) among all admissible solutions of the problem (4)–(7) if there exists a classical solution $x^*(t,\tau)$ of the following boundary value problem:*

$$L^* x^*(t,\tau) + x_t^{*\prime}(t,\tau) \in a^* \left(x^*(t,\tau); (\tilde{x}(t,\tau), L\tilde{x}(t,\tau) - \tilde{x}(t,\tau)) \right) + \partial g(\tilde{x}(t,\tau), t, \tau), \tag{10}$$

$$L\widetilde{x}(t,\tau) - \widetilde{x}'_t(t,\tau) \in a(\widetilde{x}(t,\tau), x^*(t,\tau)),$$

$$x^*(1,\tau) = 0, \tau \in G; x^*(t,\tau) = 0, \quad (t,\tau) \in H,$$

where L^* is the operator adjoint to L.

Proof By arguments analogous to those in the proof of the preceding Theorem 3.1 it is not hard to see that

$$\begin{aligned} &\int_0^1 \int_G [g(x(t,\tau),t,\tau) - g(\widetilde{x}(t,\tau),t,\tau)]\mathrm{d}t\mathrm{d}\tau \\ &\geq \int_0^1 \int_G (L^*x^*(t,\tau) + {x^*}'_t(t,\tau))(x(t,\tau) - \widetilde{x}(t,\tau))\mathrm{d}t\mathrm{d}\tau \\ &- \int_0^1 \int_G [L(x(t,\tau) - \tilde{x}(t,\tau)) - \frac{\partial}{\partial t}(x(t,\tau) - \tilde{x}(t,\tau))]x^*(t,\tau)\mathrm{d}t\mathrm{d}\tau. \end{aligned} \tag{11}$$

Using the boundary condition (7) and the condition $x^*(t,\tau) = 0$, $(t,\tau) \in H$, we get from the Green formula [24,25] that

$$\int_G [x^*(t,\tau)L(x(t,\tau) - \widetilde{x}(t,\tau)) - (x(t,\tau) - \widetilde{x}(t,\tau))L^*x^*(t,\tau)]\mathrm{d}\tau = 0. \tag{12}$$

Moreover, by the initial condition (6) and $x^*(1,\tau) = 0$ we have

$$\int_0^1 \int_G \frac{\partial}{\partial t}[x^*(t,\tau)(x(t,\tau) - \widetilde{x}(t,\tau))]\mathrm{d}t\mathrm{d}\tau = \int_G x^*(1,\tau)(x(1,\tau) - \widetilde{x}(1,\tau))\mathrm{d}\tau = 0. \tag{13}$$

Then with the use of (12), (13) we conclude that the right side of the (11) is equal to zero, that is

$$\int_0^1 \int_G g(x(t,\tau),t,\tau)\mathrm{d}t\mathrm{d}\tau \geq \int_0^1 \int_G g(\widetilde{x}(t,\tau),t,\tau)\mathrm{d}t\mathrm{d}\tau.$$

The theorem is proved.

Remark 3.1 If by analogy to the classical theory of parabolic equations we take $x'_t - Lx$ instead of $Lx - x'_t$, then a^* must be replaced by $(-a)^*$. Therefore, for the computation of $\tilde{a}^* \neq \emptyset, \tilde{a}(x) = \alpha a(x) (\alpha = \text{constant})$ we have the formula

$$\widetilde{a}^*(v^*; (x, \alpha v)) = |\alpha|\, a^*(\mathrm{sgn}\alpha; (x, v)),$$

which is easily obtained from the definition of the LCM.

Remark 3.2 With use of the Theorem 3.2 it can be obtained sufficient conditions for non-convex problem (4)–(7).

Remark 3.3 Suppose that we have the problem (4)–(7) with homogeneous boundary conditions, where $\alpha(\tau) \in L_2(G)$, $H^{1,0}(D)$ is the Hilbert space (for a more detailed study see, for example, [24,25]) consisting of the elements $x(t,\tau) \in L_2(D)$ having square-integrable generalized derivatives on D, where the inner product and the norm are defined by the respective expressions

$$\langle x_1, x_2\rangle_{H^{1,0}(D)} = \int_D (x_1 x_2 + \frac{\partial x_1}{\partial \tau}\frac{\partial x_2}{\partial \tau})\mathrm{d}t\mathrm{d}\tau, \qquad ||x||_{H^{1,0}(D)} = \sqrt{\langle x, x\rangle_{H^{1,0}(D)}}.$$

By analogy to the classical theory of the first boundary value problem for partial differential equations of parabolic type, a function $x(t,\tau) \in H^{1,0}(D)$ is called generalized solution of the problem (4)–(7) if it satisfies the boundary conditions $x(t,\tau) = 0,\ (t,\tau) \in H$ and the identity

$$\int_D \left(x\zeta_t - \sum_{i,j=1}^{n} d_{ij} \frac{\partial x}{\partial \tau_j} \frac{\partial \zeta}{\partial \tau_i} - \sum_{i=1}^{n} b_i \frac{\partial x}{\partial \tau_i}\zeta - cx\zeta \right) \mathrm{d}t\mathrm{d}\tau = \int_G \alpha\zeta(0,\tau)d\tau + \int_D f\zeta \mathrm{d}t\,\mathrm{d}\tau$$

for all $\zeta(t,\tau) \in H^1(D)$ [??] with the conditions $\zeta(1,\tau) = 0, \tau \in G, \zeta(t,\tau) = 0, (t,\tau) \in H$. Here $f = f(x)$ is an arbitrary measurable selection [23] of the multivalued mapping $a(x)$.

It easy to see that the concept of a solution almost everywhere (a.e.) can be introduced in addition to the concepts of a classical solution and generalized solution.

A function $x(t,\tau) \in H^{2,1}(D)$([24]) is said to be a solution a.e. for the problem (4)–(7) with homogeneous boundary conditions if it satisfies for almost all $(t,\tau) \in D$ the inclusion (5), the initial condition (6) and the homogeneous boundary conditions. A generalized solution for the adjoint boundary value problem is defined analogously.

4 Duality in differential inclusions of parabolic type

Let us introduce the function

$$I_*(x^*,u^*) = \iint_{\mathrm{Q}} \left[\Omega_{\mathrm{a}} \left(\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), x^*(t,\tau) \right) - g^*(u^*(t,\tau),t,\tau) \right] \mathrm{d}t\mathrm{d}\tau - \int_0^1 \langle x^*(0,\tau), \alpha(\tau)\rangle \mathrm{d}\tau.$$

Then the problem of determining the supremum

$$\sup_{\substack{x^*(t,\tau),\, u^*(t,\tau),\, x^*(1,\tau)=0,\\ x^*(t,0)=x^*(t,1)=0}} I_*(x^*,u^*) \tag{14}$$

is called the dual problem to the convex problem (1)–(3) with homogenous boundary conditions ($\beta_0(t) = \beta_1(t) = 0$). It is clear that the boundary conditions in the problem (1)–(3) can always be made homogeneous by a change of variables. We assume that $x^* \in \mathrm{C}^{1,2}(\mathrm{Q})$ and u^* is in the class of continuous functions on Q in problem (14).

Theorem 4.1 *The inequality*

$$I(x) \geq I_*(x^*,u^*)$$

is satisfied for all admissible solutions x and $\{x^*,u^*\}$ of the problem (1)–(3) with homogeneous boundary condition and all solutions of the dual problem (14), respectively.

Proof By the definition of the functions Ω_{a} and g^* it is clear that for all $x = x(t,\tau), x^* = x^*(t,\tau), u^* = u^*(t,\tau)$

$$\Omega_{\,\mathrm{a}}\Big(\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), x^*(t,\tau)\Big) \leq -\Big\langle \frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} + \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), x(t,\tau)\Big\rangle + \Big\langle x^*(t,\tau), \frac{\partial^2 x(t,\tau)}{\partial \tau^2} - \frac{\partial x(t,\tau)}{\partial t}\Big\rangle \tag{15}$$

$$- g^*(u^*(t,\tau),t,\tau) \leq g(x(t,\tau),t,\tau) - \langle x(t,\tau), u^*(t,\tau)\rangle. \tag{16}$$

Moreover using the definition $I_*(x^*, u^*)$ integrating (15),(16) and adding them we found

$$\begin{aligned} I_*(x^*(t,\tau), u^*(t,\tau)) \le & -\iint_Q \langle \frac{\partial^2 x^*(t,\tau)}{\partial \tau^2}, x(t,\tau)\rangle \mathrm{d}t\mathrm{d}\tau \\ & + \iint_Q \langle x^*(t,\tau), \frac{\partial^2 x(t,\tau)}{\partial \tau^2}\rangle \mathrm{d}t\mathrm{d}\tau \\ & - \iint_Q \frac{\partial}{\partial t}\langle x^*(t,\tau), x(t,\tau)\rangle \mathrm{d}t\mathrm{d}\tau \\ & - \int_0^1 \langle x^*(0,\tau), \alpha(\tau)\rangle \mathrm{d}\tau + I(x(t,\tau)). \end{aligned} \tag{17}$$

Similarly the proof of Theorem 3.1 using the condition $x^*(1,\tau) = 0$ it can be shown that the sum of the last two integrals on the right side of the equality (17) is equal to zero. On the other hand taking into account the homogeneous boundary conditions $x(t, 0) = 0$ and $x(t, 1) = 0$ it is not hard to show that

$$\begin{aligned} & \iint_Q [\langle x^*(t,\tau), \frac{\partial^2 x(t,\tau)}{\partial \tau^2}\rangle - \langle \frac{\partial^2 x^*(t,\tau)}{\partial \tau^2}, x(t,\tau)\rangle]\mathrm{d}t\mathrm{d}\tau \\ & = \int_0^1 [\langle \frac{\partial x^*(t,\tau)}{\partial t}, x(t,\tau)\rangle]_{\tau=0}^{\tau=1}\mathrm{d}t - \int_0^1 [\langle x^*(t,\tau), \frac{\partial x(t,\tau)}{\partial \tau}\rangle]_{\tau=0}^{\tau=1}\mathrm{d}t = 0 \end{aligned}$$

Thus we have from the inequality (17) that

$$I_*(x^*(t,\tau), u^*(t,\tau)) \le I(x(t,\tau))$$

which is what required.

Theorem 4.2 *If the solutions $\tilde{x}(t,\tau)$ and $\{x^*(t,\tau), u^*(t,\tau)\}$, $u^*(t,\tau) \in \partial g(\widetilde{x}(t,\tau), t, \tau)$ satisfy the conditions (a)–(c) of Theorem 3.1, then they are solutions of the direct and dual problems, and their values are equal.*

Proof The fact that $\widetilde{x}(t,\tau)$ is a solution of the direct problem was proved in Theorem 3.1. We study the remaining assertions. By the definition of LCM the condition (a) of Theorem 3.1 is equivalent to the inequality

$$\begin{aligned} & \langle u^*(t,\tau) - \frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} - \frac{\partial x^*(t,\tau)}{\partial t}, x - \widetilde{x}(t,\tau)\rangle \\ & + \langle x^*(t,\tau), v - \frac{\partial^2 \tilde{x}(t,\tau)}{\partial \tau^2} + \frac{\partial \tilde{x}(t,\tau)}{\partial t}\rangle \ge 0, \quad x, v \in grfa. \end{aligned}$$

This means that

$$\left(u^*(t,\tau) - \frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} - \frac{\partial x^*(t,\tau)}{\partial t}, x^*(t,\tau)\right) \in dom\Omega_a, \tag{18}$$

where

$$dom\Omega_a = \left\{\left(-x^*, v^*\right) : \Omega_a\left(x^*, v^*\right)\right\rangle - \infty\right\}$$

Moreover, since $\partial g(x,t,\tau) \subset domg^*(., t\tau)$, it is clear that $u^*(t,\tau) \in domg^*(., t, \tau)$. Then taking into account this inclusion it can be concluded from (18) that the solution $\{x^*, u^*\}$ is an admissible solution. On the other hand $u^*(t,\tau) \in \partial g(\widetilde{x}(t,\tau),t,\tau)$ is equivalent (see, for example [1,16,23]) to the relation

$$g^*(u^*(t,\tau),t,\tau) = \langle x(t,\tau), u^*(t,\tau)\rangle - g(x(t,\tau),t,\tau) \tag{19}$$

Further by [1, Lemma 2.2] we get

$$\begin{aligned}&\Omega_a\left(\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} - \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), x^*(t,\tau)\right)\\ &= W_a\left(\widetilde{x}(t,\tau), x^*(t,\tau)\right) - \langle\frac{\partial^2 x^*(t,\tau)}{\partial\tau^2} - \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), \widetilde{x}(t,\tau)\rangle.\end{aligned} \tag{20}$$

By condition (c)

$$W_a\left(\widetilde{x}(t,\tau), x^*(t,\tau)\right) = \langle\frac{\partial^2 \widetilde{x}(t,\tau)}{\partial\tau^2} - \frac{\partial \widetilde{x}(t,\tau)}{\partial t}, x^*(t,\tau)\rangle. \tag{21}$$

Then in view of (19)–(21) it is easy to establish that instead of inequality (17) in the proof of Theorem 3.1 will be equality, i.e. $I\ (\widetilde{x}(t,\tau)) = I_*\ (x^*(t,\tau), u^*(t,\tau))$ and $\{x^*(t,\tau), u^*(t,\tau)\}$ is optimal. The proof is complete.

We remark that the dual problem to the convex problem (4)–(7) with homogeneous boundary conditions consists of the following

$$\sup_{\substack{x^*(t,\tau),\, u^*(t,\tau),\, x^*(1,\tau)=0,\\ x^*(t,\tau)=0,\,(t,\tau)\in H,}} I_*(x^*,u^*), \quad \tau\in G, \tag{22}$$

where

$$\begin{aligned}I_*(x^*,u^*) = \int_0^1\int_G &\left[\Omega_a\left(L^*x^*(t,\tau) + \frac{\partial x^*(t,\tau)}{\partial t} - u^*(t,\tau), x^*(t,\tau)\right)\right.\\ &\left.-g^*\left(u^*(t,\tau),t,\tau\right)\right]\mathrm{d}t\mathrm{d}\tau - \int_G x^*(0,\tau)\alpha(\tau)\mathrm{d}\tau.\end{aligned} \tag{23}$$

By extending the proofs of Theorems 4.1 and 4.2 to the case under consideration it is not hard to get the following result.

Theorem 4.3 *If* $\widetilde{x}(t,\tau)$ *and* $\{x^*(t,\tau), u^*(t,\tau)\}$, $u^*(t,\tau) \in dom g^*(.,t,\tau)$ *are admissible solutions to the direct convex problem* (4)–(7) *with homogeneous boundary conditions and of the dual problem* (22), (23)*, respectively, then*

$$I_*(\widetilde{x}(t,\tau)) \geq I_*\left(x^*(t,\tau), u^*(t,\tau)\right).$$

If the condition of Theorem 3.3 that suffices for optimality is valid here, then equality holds, and $\{x^*(t,\tau), u^*(t,\tau)\}$ is a solution of the dual problem.

In the conclusion of this section we consider an example:

$I\ (x) \to \inf$,

where

$$\frac{\partial^2 x(t,\tau)}{\partial\tau^2} - \frac{\partial x(t,\tau)}{\partial t} = Ax(t,\tau) + Bu(t,\tau), \quad u(t,\tau) \in \mathrm{U}$$

$$x\,(0,\tau) = \alpha\,(\tau)\,, x(t,0) = \beta_0(t), \quad x(t,1) = \beta_1(t) \tag{24}$$

where A and B are $n \times n$ and $n \times r$ matrices, respectively, $U \subset R^r$ is a convex closed set, and g is continuously differentiable function of x. It is required to find a controlling parameter $\widetilde{u}(t, \tau) \in U$ such that the solution $\widetilde{x}(t, \tau)$ corresponding to it minimizes I.
In the considered case

$$a(x) = Ax + BU. \tag{25}$$

It is not hard to see that

$$a^*(v^*; (x, v)) = \begin{cases} A^*v^*, & B^*v^* \in K_U^*(\tilde{u}), \\ \emptyset, & B^*v^* \notin K_U^*(\tilde{u}), \end{cases}$$

where $v = Ax + B\widetilde{u}, \widetilde{u} \in U$.

Then applying Theorem 3.1 we get

$$\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} - \frac{\partial x^*(t,\tau)}{\partial t} = A^*, x^*(t,\tau) + g'(\tilde{x}(t,\tau), t, \tau),$$

$$x^*(1,\tau) = 0, \quad x^*(t,0), \quad x^*(t,1) = 0,$$

$$\langle B\widetilde{u}(t,\tau), x^*(t,\tau)\rangle = \inf_{u \in U} \langle Bu, x^*(t,\tau)\rangle. \tag{26}$$

Therefore, we have obtained the following theorem.

Theorem 4.4 *The solution $\tilde{x}$ corresponding to the control $\tilde{u}$ minimizes I in the problem* (24) *if there exists a function x^* satisfying the condition* (26)*.*

On the other hand by elementary computations we find that

$$\Omega_a\left(x^*, v^*\right) = \begin{cases} -\infty, & x^* \neq A^*v^*, \\ -W_U, & x^* = A^*v^*. \end{cases}$$

Thus in the dual problem (24) $I_*(x^*, u^*)$ has the form:

$$I_*(x^*, u^*) = -\iint_Q [W_U\left(-B^*x^*(t,\tau)\right) + g^*\left(u^*(t,\tau), t, \tau\right)]\mathrm{d}t\mathrm{d}\tau - \int_0^1 \langle x^*(0,\tau), \alpha(\tau)\rangle \mathrm{d}\tau,$$

where

$$\frac{\partial^2 x^*(t,\tau)}{\partial \tau^2} - \frac{\partial x^*(t,\tau)}{\partial t} = A^*x^*(t,\tau) + u^*(t,\tau)$$

and W_U is a support function of U.

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